Uncertainty Relation in Flat Space from Holography

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Abstract

Following the idea that information is the negative entropy, we propose that the information and entropy of an isolated system can convert into each other while the sum of them is an invariant for any physical process. The holographic principle is then reformulated in the way that this invariant is bounded by the Bekenstein-Hawking entropy of the system. It is found that Heisenberg's uncertainty relation in quantum mechanics can be derived from this bound.

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The concepts of information and entropy [1, 2] play more and more important roles in physics both in technical and theoretical aspects [3]. It seems that information is more fundamental than space as well as matter since each physical law must be written in terms of a certain type of information such as coordinate, curvature, energy, mass, etc.. Hence, in order to construct the unified law of all interactions in nature, one should inquire what are the informational contexts of space and matter and how to measure these physical quantities in a unified framework of information. Ever since the holographic principle was proposed [4, 5] the information and entropy manifest themselves as key concepts as a guide for constructing successful theory of quantum gravity (for recent review, see [6]). The information, known as how much uncertainty can be eliminated in light of an observer, can be defined as the deviation of the actual entropy of a system from the maximal entropy that the system may contain [7]. The feature of information is that an observer's acquiring of information about an isolated system is always associated with decreasing of uncertainty on the system. Since information describes the observer's capability to predict the outcome of physical system some time later, it is an observer-dependent quantity.

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However, the sum of the information and entropy may be of physically significant if this sum is independent of the observers.

In this Letter, we argue that information and entropy of a system are two sides of one coin in the sense that they can convert into each other in the process of measurement and evolution of the system while the sum of them is an invariant. We show that this invariant is the maximal information (or entropy) by which the holographic principle is reformulated. By applying this assumption to a system of one particle, we derive uncertainty relation from the holographic principle.

Let p be the probability of obtaining the information I for an observer. The relation between them can be given by the fact that probability is multipliable while information is additive:

$$I = k \ln(1/p). \tag{1}$$

This is the well-known Shannon formula for information. By setting $k = 1/\ln 2$, one finds that Eq. (1) becomes

$$I = \log_2(1/p). \tag{2}$$

This implies $I = 1$ when $p = 1/2$, that is, the information to make decision in two possible choices with equal probability is exactly 1 bit.

Since the probabilities p_i of acquiring the different information I_i vary or depend on the contexts of the information I_i, $i = 1, \cdots, N$, the total amount of information should be the direct sum of $I_i : I = \sum_{i=1}^{N} I_i$. The average information $\overline{I}$ is

$$\overline{I} = \sum_{i=1}^{N} p_i I_i = -\sum_{i=1}^{N} p_i \log_2 p_i. \tag{3}$$

It is evident that in the case of equal probabilities $p_i = p$ for all p_i

$$\overline{I} = -\log_2 p = -k \ln p = I_i. \tag{4}$$

This enable us to look the i-th information I_i as average information.

We know that entropy is the measurement of how much a system is in disorder while I is exactly the opposite. For this reason, one can define the information of an observer as the deviation of the actual entropy from the maximal entropy [7]. Clearly, this concept is well defined for an isolated system only if we assume that the maximal entropy is fixed. Motivated by the general consideration of the relationship between information and entropy in physics and beyond, we propose an ansatz that the sum of entropy and information (or, the total IE) is an invariant for an isolated system:

$$S + I = Invariant. \tag{5}$$

in which the constant keep invariant during the system evolution. This implies that for an observer the entropy S of an isolated system will decrease as information I about it increases, and vice versa. In this sense, information is nothing but IE though it may have not become into knowledge about the system for a specified observer. In the following, we will provide a demonstration of why this is so.

According to the standard thermodynamics, the entropy S of an isolated system is given by the Boltzman formula

$$S = k_B \ln W, \tag{6}$$

in which W is the total number of the microscopic states corresponding to a macroscopic state of the system. Notice that p in Eq. (1) is the probability of the macro-state appearing as an information, one has $p = W \cdot p_0$, where p_0 is the probability of a micro-state appearing. Hence, if we choose unified unit for information and entropy in (1) and (6) so that $k_B = k$, one finds

$$k_B \ln W + k \ln(1/p) = k \ln(1/p_0) = I_0. \tag{7}$$

For an isolated system with definite volume, N is the total number of the elementary particles, which is fixed. Since volume element L in phase space is finite and definite, $p_0 = 1/L^N$ is also definite. This shows that Eq. (5) holds for the system.

We rewrite Eq. (7) as

$$S = I_0 - I. \tag{8}$$

Then when an observer finally acquires, by repeating measurements enough times, maximal amount of information $I_{\max}$ about a given system, one finds $I_0 = I_{\max}$, that is

$$S = I_{\max} - I. \tag{9}$$

Eq. (9) indicates that entropy is the measurement of a part of information that the observer is short of in trying to accurately describe the states of system.

Now that the conservation equation (5) should hold for any physical system on matter what underlying interactions involved, it must be true for a black hole. For a black hole with horizon area A we know from the well-known Bekenstein-Hawking formula [8, 9, 10] that its entropy is given by

$$S_{BH} = \frac{k_B A}{4 l_p^2}, \tag{10}$$

in which $l_p = (G\hbar/c^3)^{1/2}$ stands for the Planck length. In the Planck unit this entropy is a quarter of the area of its horizon. Since we know nothing about the black hole ($I_{BH} = 0$), it follows from Eq. (5) that

$$S_{BH} = I_{\max}. \tag{11}$$

This implies, together with Eq. (10), that the information of an outer observer about a black hole has completely converted into entropy proportional to its horizon area. In other words, one can reinterpret Eq. (11), by the assumption (5), as that the information has all been saved on the horizon boundary of the black hole as its entropy. It is clear that Eq. (10) is the limit case of Eq. (5) in above sense. Now that Eq. (10) is believed to be valid without depending the dynamical details of interactions [6, 11, 12], one can, in principle, assume that it applies for any system with strong gravitational interaction, including the system of photons.

If one inputs energy from outer region into the system through its boundary, the IE within the system will increase. Then, Eq. (11) indicates that the total IE of an isolated system(or space-time region) with boundary area A is bounded up by the entropy S_{BH} of a black hole with same area of horizon,

$$S + I \leq S_{BH} = \frac{k_B A}{4 l_p^2}. \tag{12}$$

Clearly, this agrees with the holographic principle $S \leq A/4$ in Planck unit since information I of an observer about the system can not be negative. The right side of Eq. (12) is known as the holographic bound [6]. We know that holographic principle provides for a system an upper bound of information of 1 bit per Planck area $A_p = (2l_p)^2$ on its boundary. We note here that Eq. (12) has reformulated the holographic principle by division the IE into two parts S and I explicitly.

To see the implication of Eq. (12), we reconsider the case of black hole. From Eq. (9) we know that, in order to acquire information from black hole, it is necessary for the outer observer to reduce its entropy, or, equivalently, reduce the area of its horizon. This can be exactly done by the process of Hawking radiation [13]. Hence, in light of the formulation (12), the Hawking process will necessarily carry the information out of the black hole which has previously disappeared in it. The black hole information paradox is then eliminated.

Let us consider, for simplicity, a free particle with mass M. According to the holographic bound (12) there exists a space-time region V enclosing this particle such that the IE of the particle would not be more than that given by the area A of the boundary ∂V of V. Holographic principle implies that V can not be a geometric point since otherwise the IE of this particle will be equal or less that the entropy bound $0 = A/4$, which contradicts with the fact that it has the nonzero IE. In addition, a point-like particle can not be possible since otherwise it appears as a singularity in space-time, which must be surrounded by an even horizon (∂V) to avoid a bare singularity. The radius of this region is the lower limit for position observation. Therefore, one has following inequality for the

entropy S of the particle

$$S \leq \frac{k_B A}{4 l_p^2}. \tag{13}$$

Even if we know nothing about the makeup of the particle we can assume that the energy $E = Mc^2$ of the particle arises from the contributions of its internal degrees of freedom constrained by the bound (13). We can then associate the energy E with a temperature T which corresponds to some unknown motion of these degrees of freedom. This temperature T should be in consistent with the standard relation of entropy in thermodynamics: $S = \int_0^E dE/T$. Then, it follows from Eq. (13) that

$$\frac{Mc^2}{T} \leq \frac{k_B A}{4 l_p^2},$$

or,

$$Mc^2 \leq k_B T \cdot \frac{A}{4 l_p^2}. \tag{14}$$

For the internal motion of all degrees of freedom within one-particle system, one can estimate that the energy of this system is about $Mc^2 \approx k_B T$. Therefore, Eq. (14) becomes

$$\frac{A}{4 l_p^2} \geq 1,$$

or, using $l_p^2 = G\hbar/c^3$,

$$\frac{4\pi R^2}{4G\hbar/c^3} = \frac{\pi R^2 c^3}{G\hbar} \geq 1, \tag{15}$$

in which R is the characteristic radius of the region V. For an observer who it is going to measure this particle V is the hidden region and the boundary $A = 4\pi R^2$ is the horizon area of the particle with mass M. Thus, R should be given by Schwarzschild radius [14]: $R = 2MG/c^2$. If we rewrite Eq. (15) as

$$\frac{Rc^2}{2G} \cdot \frac{2\pi c R}{\hbar} \geq 1.$$

Then, one finds

$$RMc \geq \frac{\hbar}{2\pi}. \tag{16}$$

The fact that region V is hidden for the observer outside implies that the implication of the radius R can be understood as the lower limit of the position measurement implemented by the observer. Therefore, one can identify R as the position uncertainty $R = \Delta x$ of the particle with respect to the observer. We also know [14] that the hidden

region V given by Schwarzschild radius is the one that can not be measured even by the fastest particles, namely, the photons. Therefore, $Mc = Mc^2/c$ is the most-likely loss of the particle momentum during the measuring process since the momentum transfer is maximal for the measurable particle with vanishing rest mass. Hence, $Mc \approx \Delta p$. So, it follows from Eq. (16) that

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2\pi}. \tag{17}$$

This is the well-known uncertainty relation in quantum mechanics up to a constant of order $1/\pi$. In spite of the presence of the mystical factor $1/\pi$ in the relation (17) it is in consistent with the Heisenberg's relation $\Delta x \Delta p \geq \hbar/2$. Since the validity of Heisenberg's relation is mainly confined in the phenomena with the gravity relatively weak in contrast with other interactions, one can look Eq. (17) as the uncertainty relation that may generally holds in quantum gravity.

In summary, we proposed an ansatz that the sum of information and entropy of an isolated system is identically conserved in the process of measurement and evolution of the system, by which the holographic principle was reformulated in terms of the concept of the information entropy. It was found that uncertainty principle between position and momentum can be derived from the holographic principle. Therefore, the uncertainty relation in quantum theory is a manifestation of the finiteness of total information that an observer can acquire by measuring a system.

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